Diameter-girth sufficient conditions for optimal extraconnectivity in graphs*

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Abstract

For a connected graph $G$, the $r$-th extraconnectivity $\kappa_r(G)$ is defined as the minimum cardinality of a cutset $X$ such that all remaining components after the deletion of the vertices of $X$ have at least $r+1$ vertices. The standard connectivity and superconnectivity correspond to $\kappa_0(G)$ and $\kappa_1(G)$, respectively. The minimum $r$-tree degree of $G$, denoted by $\xi_r(G)$, is the minimum cardinality of $N(T)$ taken over all trees $T \subseteq G$ of order $|V(T)| = r + 1$, $N(T)$ being the set of vertices not in $T$ that are neighbors of some vertex of $T$. When $r = 1$, any such considered tree is just an edge of $G$. Then, $\xi_1(G)$ is equal to the so called minimum edge-degree of $G$, defined as $\xi(G) = \min\{d(u) + d(v) - 2 : uv \in E(G)\}$, where $d(u)$ stands for the degree of vertex $u$. A graph $G$ is said to be optimally $r$-extraconnected, for short $\kappa_r$-optimal, if $\kappa_r(G) \geq \xi_r(G)$. In this paper, we present some sufficient conditions that guarantee $\kappa_r(G) \geq \xi_r(G)$ for $r \geq 2$. These results improve some previous related ones, and can be seen as a complement of some others which were obtained by the authors for $r = 1$.

Keywords: Connectivity, superconnectivity, cutset, diameter, girth.

*This work has been published in Discrete Math., 308(16) (2008) 3526–3536.
†This research was supported by the Ministry of Science and Innovation, Spain, and the European Regional Development Fund (ERDF) under project MTM2005-08990-C02-02
1 Introduction

Throughout this paper, only undirected simple graphs without loops or multiple edges are considered. Unless otherwise stated, we follow the book by Chartrand and Lesniak [8] for terminology and definitions.

Let \( G = (V, E) \) be a connected graph with the set of vertices \( V(G) = V \) and the edge set \( E(G) = E \). If \( d_G(x, y) = d(x, y) \) stands for the distance between vertices \( x \) and \( y \), then the distance between subsets of vertices \( S \) and \( T \) is just \( d_G(S, T) = \min \{d(x, y) : x \in S, y \in T\} \); we write \( d(x, T) \) instead of \( d(\{x\}, T) \). For every \( S \subset V \) and every nonnegative integer \( r \geq 0 \), \( N_r(S) \) denotes the set \( \{z \in V : d(z, S) = r\} \); additionally, \( N_1(S) \) will be written as \( N(S) \), and \( N_r(\{x\}) \) will be simplified to \( N_r(x) \) for every \( x \in V \). The subgraph of \( G \) induced by \( S \subset V \) is written \( G[S] \).

A subset \( X \) of vertices is said to be a cutset if \( G - X \) is not connected. A cutset is called a \( P_r \)-cutset if every component of \( G - X \) has at least \( r + 1 \) vertices. If \( G \) has at least one \( P_r \)-cutset, the \( r \)-th extraconnectivity of \( G \), denoted by \( \kappa_r(G) \), is then defined as the minimum cardinality over all \( P_r \)-cutsets of \( G \) [1, 10, 11, 13]. From the definition, we immediately have that if \( \kappa_r(G) \) exists, then \( \kappa_i(G) \) exists for any \( i < r \) and \( \kappa_i(G) \leq \kappa_r(G) \). Obviously, every cutset of \( G \) is a \( P_0 \)-cutset, and \( \kappa_0(G) \) is just the standard connectivity \( \kappa(G) \). It is widely known that \( \kappa(G) \leq \delta(G) \), where \( \delta(G) \) is the minimum degree of \( G \). Hence a graph \( G \) is called maximally connected if \( \kappa(G) = \delta(G) \). The first extraconnectivity \( \kappa_1(G) \) has been studied under the name of superconnectivity. This is a stronger measure of connectivity than the standard connectivity, and was first proposed in [5, 6]. The corresponding index for edges, \( \lambda'(G) \), is called restricted edge-connectivity and was proposed by Esfahanian and Hakimi [9]. The study of this parameter has been approached in several articles (see, for instance, [2, 4, 7, 14, 15, 16]). A graph is superconnected, for short super-\( \kappa \), if every minimum cutset consists of a set of vertices that are all of them adjacent to one vertex which does not belong to the cutset, see Boesch [5], Boesch and Tindell [6] and Fiol, Fàbrega and Escudero [12]. Observe that a superconnected graph is necessarily maximally connected, but the converse is not true (take \( C_g \) — a cycle of length \( g \) — with \( g \geq 6 \) as a simple example of a maximally connected graph that is not superconnected). Notice also that \( \kappa_1(G) > \delta(G) \) is a sufficient and necessary condition for \( G \) be superconnected.

The minimum edge-degree of \( G \) is \( \xi(G) = \min \{d(u) + d(v) - 2 : uv \in E(G)\} \), \( d(u) \) standing for the degree of vertex \( u \). A graph \( G \) is said to be optimally superconnected if \( \kappa_1(G) \geq \xi(G) \). As a generalization of \( \xi(G) \) we define the minimum \( r \)-tree-degree of \( G \), denoted by \( \xi_r(G) \), as follows:

\[
\xi_r(G) = \min \left\{ \sum_{v \in V(T)} d(v) - 2r : T \subseteq G \text{ is a tree of order } r + 1 \right\}.
\]

Clearly, \( \xi_1(G) = \xi(G) \). A connected graph \( G \) is said to be \( \kappa_r \)-connected if \( \kappa_r(G) \) exists. We will show that \( G \) is a \( \kappa_r \)-connected graph with \( \kappa_r(G) \leq \xi_r(G) \) when the girth is \( g \geq r + 5 \) and the minimum degree is \( \delta(G) \geq 3 \). A \( \kappa_r \)-connected graph \( G \) is said to be optimally \( r \)-extraconnected, for short \( \kappa_r \)-optimal, if \( \kappa_r(G) \geq \xi_r(G) \).

Some sufficient conditions to guarantee lower bounds for the \( r \)-th extraconnectivities, \( r \geq 1 \),
have been given in [1, 3, 10, 11]. Some of the results contained in these references are listed below in chronological order.

**Theorem 1.1** Let $G$ be a graph with minimum degree $\delta \geq 2$, girth $g$ and diameter $D$.

(i) [10, 11] Let $r \geq 2$ and $\delta \geq 3$. Then $\kappa_r(G) \geq (r+1)\delta - 2r$ if $D \leq 2\left\lceil\frac{(g-1)/2 - r - 1}{r} \right\rceil$ for $r$ even, and if $D \leq 2\left\lceil\frac{(g-1)/2 - r - 2}{r} \right\rceil$ for $r$ odd.

(ii) [1] Let $r \geq 3$ and $\delta \geq 3$. Then $\kappa_r(G) \geq (r+1)\delta - 2r$ if

- $D \leq 2\left\lceil\frac{(g-1)/2 - 5}{r} \right\rceil$ and $\delta \geq r - 2$;
- $D \leq 2\left\lceil\frac{(g-1)/2 - 7}{r} \right\rceil$ and $\delta \geq \lceil(r-1)/2\rceil$.

(iii) [3] $\kappa_1(G) \geq \xi(G)$ if $D \leq g - 3$.

In this paper we improve the results (i) and (ii) of Theorem 1.1. Among other results we will show that: $\kappa_2(G) = \xi_2(G)$ if $D \leq g - 4$, when $\delta \geq 3$ and $g \geq 7$; and, for all $r \geq 3$, $\kappa_r(G) = \xi_r(G)$ if $D \leq g - 7$, when $\delta \geq \max\{3, \lceil(r+1)/2\rceil\}$ and $g \geq r + 5$.

## 2 Main Results

We start by presenting a sufficient condition that assures the existence of $P_r$-cutsets in a connected graph.

**Lemma 2.1** Let $r \geq 1$ be an integer and let $G$ be a connected graph with girth $g \geq r + 5$ and minimum degree $\delta \geq 3$. Then $G$ is $\kappa_r$-connected and $\kappa_r(G) \leq \xi_r(G)$.

**Proof:** Clearly $G$ contains some cycle, because $\delta \geq 3$. Hence, $|V(G)| > g \geq r + 5$, and we can consider in $G$ a tree $T$ on $r + 1$ vertices. Let $X \subset V(G)$ be the neighborhood of $T$; that is, $X = N(T)$. As the diameter of $T$ is $D(T) \leq r$ and $g \geq r + 5$, it follows that the induced subgraph $G[V(T) \cup X]$ is acyclic. Then, $G \neq G[V(T) \cup X]$, which implies that $X$ is a cutset of $G$ and $T$ is a component of $G - X$.

Let us show that $X$ is a $P_r$-cutset. We reason by contradiction, supposing that there exists some component $C$ of $G - X$ with less than $r + 1$ vertices. Observe that $|V(C)| \leq r$ means that $C$ is a tree, hence there exists some vertex $z \in V(C)$ such that $d_C(z) \leq 1$ (inequality only in case $|V(C)| = 1$). As a consequence, vertex $z$ must be adjacent to at least two vertices of $X$, say $z_1$, $z_2$, because $\delta \geq 3$. Denoting by $t_1, t_2 \in V(T)$ to some vertices such that $z_1t_1$ and $z_2t_2$ are edges of $G$, we have that $zz_1t_1 \ldots t_2z_2z$ is a cycle in $G$, $t_1 \ldots t_2$ being a path in $T$. But the length of this cycle is $2 + d_T(t_1, t_2) + 2 \leq r + 4$ since $d_T(t_1, t_2) \leq D(T) \leq r$, contradicting the fact that $g \geq r + 5$.

Finally, $\kappa_r(G) \leq \xi_r(G)$ follows if we choose a tree $T$ so that $|X| = |N(T)| = \xi_r(G)$. $\blacksquare$

Notice that this lemma does not hold for $\delta = 2$, unless other additional conditions are imposed on the graph (see [3] for the case $r = 1$). For instance, even though $g \geq r + 5$ holds
for the cycle $C_{2r+3}$ when $r \geq 2$, is is quite simple to see that $C_{2r+3}$ does not have $\mathcal{P}_r$-cutsets, because the graph is 2-connected.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{Sets of vertices related to a tree $T$ and a $\mathcal{P}_r$-cutset $X$.}
\end{figure}

At this point we need to introduce some more notation. Let $X \subset V$ be a $\mathcal{P}_r$-cutset of $G$ and let $C$ be a connected component of $G - X$. For a given integer $r \geq 1$, let $T$ be a tree of order $r + 1$ contained in $C$ whose set of vertices is denoted by $V(T) = \{v_1, \ldots, v_{r+1}\}$. For each $v_i \in V(T)$, we define the following sets (see Fig 1):

\begin{align*}
S^+_i &= \{ z \in N(v_i) \setminus V(T) : d(z, X) = d(v_i, X) + 1 \} \\
S^=_i &= \{ z \in N(v_i) \setminus V(T) : d(z, X) = d(v_i, X) \} \\
S^-_i &= \{ z \in N(v_i) \setminus V(T) : d(z, X) = d(v_i, X) - 1 \} \\
N^+_i &= \{ z \in N(S^+_i) : d(z, X) = d(v_i, X) + 2 \} \\
N^-_i &= \{ z \in N(S^+_i) : d(z, X) = d(v_i, X) + 1 \} \\
N^-_i &= \{ z \in N(S^+_i) - v_i : d(z, X) = d(v_i, X) \}
\end{align*}

Given a $\mathcal{P}_r$-cutset $X$ of a graph $G$, for any fixed connected component $C$ of $G - X$ let us denote by $\mu(C) = \max\{d(u, X) : u \in V(C)\}$. When the referred component is clear we only write $\mu$ instead of $\mu(C)$. Then $N_\mu(X) \cap V(C) = \{ u \in V(C) : d(u, X) = \mu \}$. We present the following lemmas proving some useful lower bounds on $\mu$ depending on the properties of the induced subgraph $G[N_\mu(X) \cap V(C)]$.

**Lemma 2.2** Let $r \geq 2$ be an integer and let $G$ be a $K_r$-connected graph with girth $g \geq r + 5$ and minimum degree $\delta \geq \lceil (r + 1)/2 \rceil$. Let $X$ be a minimum $\mathcal{P}_r$-cutset of $G$, and let $C$ be any connected component of $G - X$. If $G$ is non $K_r$-optimal, then:

(i) $\mu = \max\{d(u, X) : u \in V(C)\} \geq 2$.

(ii) If there exists some edge $uv$ in $C$ such that $d(u, X) = d(v, X) = \mu$, then
\[ \mu = \max\{d(u, X) : u \in V(C)\} \geq \begin{cases} 
\lfloor (g - 6)/2 \rfloor & \text{if } r \geq 3; 
\lfloor (g - 3)/2 \rfloor & \text{if } r = 2. 
\end{cases} \]

**Proof:** (i) Clearly, \( \mu = \max\{d(u, X) : u \in V(C)\} \geq r + 1 \) if \( r + 1 \) there exists in \( C \) a tree \( T \) of order \( r + 1 \), whose set of vertices is denoted by \( V(T) = \{v_1, \ldots, v_{r+1}\} \). Consider the sets defined in (1) and notice that \( S_i^+ = \emptyset, S_i^- = N(v_i) \cap X \) and \( S_i^- = (N(v_i) \setminus V(T)) \cap V(C) \), for \( i = 1, \ldots, r+1 \). Observe that the sets \( S_i^- \), \( N(S_i^-) \cap X \) for \( i = 1, \ldots, r+1 \) are mutually disjoint, because otherwise a cycle of length at most 2 exists. Likewise, by the Pigeonhole Principle, \( |N(S_i^-) \cap X| \geq |S_i^-| \) for each \( i = 1, \ldots, r+1 \) as \( g \geq 5 \) and every vertex in \( C \) is adjacent to some vertex in \( X \) because \( \mu = 1 \). Then:

\[
|X| \geq \sum_{i=1}^{r+1} (|S_i^-| + |N(S_i^-) \cap X|) \geq \sum_{i=1}^{r+1} (|S_i^-| + |S_i^-|)
\]

\[
= \sum_{i=1}^{r+1} |N(v_i) \setminus V(T)| = \sum_{i=1}^{r+1} d(v_i) - 2r \geq \xi_r(G),
\]

and this is an absurdity because \( |X| = \kappa_r(G) < \xi_r(G) \) by hypothesis, hence \( \mu \geq 2 \).

(ii) First, suppose \( r \geq 3 \) and reason by contradiction supposing that \( 2 \leq \mu \leq \lfloor (g - 6)/2 \rfloor - 1 \), so \( g \geq 12 \). By hypothesis we can take and edge \( uv \) in \( C \) such that \( d(u, X) = d(v, X) = \mu \). Notice that \( G[N(u) \cup N(v)] \) is a subgraph of \( C \) because \( \mu \geq 2 \). Moreover, \( |N(u) \cup N(v)| \geq r + 1 \), since \( \delta \geq \lceil (r + 1)/2 \rceil \) and \( g \geq 4 \). Hence, \( G[N(u) \cup N(v)] \) contains a tree \( T \) of order \( r + 1 \) and diameter at most three. Suppose that the set of vertices \( V(T) = \{v_1, v_2, \ldots, v_{r+1}\} \) is such that \( v_1 = u, v_2 = v, d(v_1, X) = \mu \), for \( i = 1, \ldots, s \) and \( d(v_i, X) = \mu - 1 \), for \( i = s + 1, \ldots, r+1 \). Then the following sets

\[ N_{\mu-1}(S_i^-) \cap X \text{ and } N_{\mu}(S_i^+) \cap X \text{ (for } i = 1, 2), \]

\[ N_{\mu}(v_i) \cap X \text{ and } N_{\mu}(S_i^-) \cap X \text{ (for } i = 3, \ldots, s), \]

\[ N_{\mu-1}(v_i) \cap X, N_{\mu-1}(S_i^-) \cap X, N_{\mu}(N_i^+) \cap X \text{ and } N_{\mu-1}(N_i^-) \cap X \text{ (for } i = s + 1, \ldots, r+1), \]

are mutually disjoint, because otherwise a cycle of length at most \( 2\mu + 7 \leq 2(\lfloor (g - 6)/2 \rfloor - 1) + 7 \leq g - 1 \) would be found. Moreover, by the same reason,

\[ |N_{\mu-1}(S_i^-) \cap X| \geq |S_i^-| \text{ and } |N_{\mu}(S_i^+) \cap X| \geq |S_i^+| \text{ (for } i = 1, 2), \]

\[ |N_{\mu}(v_i) \cap X| \geq |S_i^-| \text{ and } |N_{\mu}(S_i^-) \cap X| \geq |S_i^-| \text{ (for } i = 3, \ldots, s), \]

\[ |N_{\mu-1}(v_i) \cap X| \geq |S_i^+|, |N_{\mu-1}(S_i^-) \cap X| \geq |S_i^-| \text{ and } \]

\[ |N_{\mu}(N_i^+) \cap X| + |N_{\mu-1}(N_i^-) \cap X| \geq |S_i^+| \text{ (for } i = s + 1, \ldots, r+1). \]
Hence,

\[ |X| \geq \sum_{i=1}^{2} (|N_{\mu-1}(S_i^-) \cap X| + |N_{\mu}(S_i^+) \cap X|) \]

\[ + \sum_{i=3}^{r+1} (|N_{\mu}(v_i) \cap X| + |N_{\mu}(S_i^+) \cap X|) \]

\[ + \sum_{i=s+1}^{r+1} (|N_{\mu-1}(v_i) \cap X| + |N_{\mu-1}(S_i^+) \cap X| + |N_{\mu}(N_{\mu-1}(S_i^-) \cap X| + |N_{\mu-1}(N_{\mu-1}(S_i^-) \cap X|) \]

\[ \geq \sum_{i=1}^{3} (|S_i^-| + |S_i^+|) + \sum_{i=s+1}^{r+1} (|S_i^-| + |S_i^+| + |S_i^+|) = \sum_{i=1}^{r+1} d(v_i) - 2r \geq \xi_r(G), \]

a contradiction.

Second, suppose \( r = 2 \) and reason again by contradiction supposing that \( 2 \leq \mu \leq \lfloor (g - 3)/2 \rfloor - 1 \), thus \( g \geq 9 \). First assume that \( C \) contains a path \( v_1v_2v_3 \) of length 2 such that \( d(v_i, X) = \mu, i = 1, 2, 3 \). Then the sets \( N_{\mu}(v_i) \cap X \) and \( N_{\mu}(S_i^+) \cap X \), for \( i = 1, 2, 3 \) are mutually disjoint, because cycles of length at most \( 2\mu + 4 \leq 2 \lfloor (g - 3)/2 \rfloor - 1 + 4 \leq g - 1 \) are forbidden. By the same reason, we have \( |N_{\mu}(v_i) \cap X| \geq |S_i^-| \) and \( |N_{\mu}(S_i^+) \cap X| \geq |S_i^+| \), for \( i = 1, 2, 3 \), hence

\[ |X| \geq \sum_{i=1}^{3} (|N_{\mu}(v_i) \cap X| + |N_{\mu}(S_i^+) \cap X|) \]

\[ \geq \sum_{i=1}^{3} (|S_i^-| + |S_i^+|) = 3 \sum_{i=1}^{3} d(v_i) - 4 \geq \xi_2(G), \]

against our assumptions. Finally, suppose that only isolated edges \( v_1v_2 \) with \( d(v_i, X) = \mu, i = 1, 2 \), are contained in \( C \), and consider the path \( v_1v_2v_3 \) in \( C \), where \( v_3 \in N(v_2) - v_1 \). Clearly, \( d(v_3, X) = \mu - 1 \). In this case, any two sets of \( N_{\mu-1}(S_i^-) \cap X \), for \( i = 1, 2, 3 \), \( N_{\mu-1}(S_i^-) \cap X \), \( N_{\mu-1}(N_{\mu-1}(S_i^-) \cap X \) and \( N_{\mu-1}(N_{\mu-1}(S_i^-) \cap X \) are disjoint, since there exist no cycles of length at most \( 2\mu + 4 \leq g - 1 \). Further, \( |N_{\mu-1}(S_i^-) \cap X| \geq |S_i^-| \), for \( i = 1, 2, 3 \), \( |N_{\mu-1}(v_3) \cap X| \geq |S_3^-| \), and \( |N_{\mu}(N_{\mu-1}(S_3^-) \cap X| + |N_{\mu-1}(N_{\mu-1}(S_3^-) \cap X| \geq |S_3^+| \) follow taking into account the lower bound for the girth of \( G \) and that \( \delta \geq 0 \) holds for minimum degree. Hence, as \( S_i^- = S_i^+ = S_2^- = S_2^+ = \emptyset \), we deduce

\[ |X| \geq \sum_{i=1}^{2} (|N_{\mu-1}(S_i^-) \cap X| + |N_{\mu-1}(v_3) \cap X| + |N_{\mu-1}(S_3^-) \cap X| \]

\[ + |N_{\mu}(N_{\mu-1}(S_i^-) \cap X| + |N_{\mu-1}(N_{\mu-1}(S_i^-) \cap X| \]

\[ \geq |S_i^-| + |S_2^-| + |S_3^-| + |S_2^+| + |S_3^+| = d(v_1) + d(v_2) + d(v_3) - 4 \geq \xi_2(G), \]

that is absurd.

From these absurdities for cases \( r \geq 3 \) and \( r = 2 \) we deduce the claimed lower bound for \( \mu = \max\{d(u, X) : u \in V(C)\} \).

\begin{lemma} \textit{Let} \( r \geq 2 \) \textit{be an integer and let} \( G \) \textit{be a} \( \kappa_r \)-
\textit{connected graph with girth} \( g \geq r + 5 \) \textit{and minimum degree} \( \delta \geq \lfloor (r + 1)/2 \rfloor \). \textit{Let} \( X \) \textit{be a minimum} \( \mathcal{P}_r \)-cutset of \( G \), \textit{and let} \( C \) \textit{be any} \end{lemma}
connected component of \( G - X \). If \( G \) is non \( \kappa_r \)-optimal and \( G[\mu(X) \cap V(C)] \) does not contain any edge, then

\[
\mu = \max\{d(u, X) : u \in V(C)\} \geq \begin{cases} 
\lfloor (g - 5)/2 \rfloor & \text{if } r \geq 3; \\
\lfloor (g - 1)/2 \rfloor & \text{if } r = 2.
\end{cases}
\]

**Proof:** By Lemma 2.2 (i) it follows \( \mu \geq 2 \). We reason by contradiction supposing that \( 2 \leq \mu \leq \lfloor (g - 5)/2 \rfloor - 1 \), so \( g \geq 11 \), if \( r \geq 3 \); and \( 2 \leq \mu \leq \lfloor (g - 1)/2 \rfloor - 1 \), hence \( g \geq 7 \), if \( r = 2 \). Since \( G[\mu(X) \cap V(C)] \) has no edges, then \( d_C(x, y) \geq 2 \) for every two vertices \( x, y \) in \( \mu(X) \cap V(C) \). First, suppose that there exist two vertices \( z, t \in \mu(X) \cap V(C) \) such that \( d_C(z, t) = 2 \). Clearly, \( G[\{z, t\} \cup N(z) \cup N(t)] \) is a subgraph of \( C \), hence \( C \) contains a tree \( T \) (with \( z, t \in V(T) \)) on \( r + 1 \) vertices because \( \delta \geq \lceil (r + 1)/2 \rceil \) and \( g \geq 4 \). Further, the diameter of \( T \) is \( D(T) = 2 \) for \( r = 2 \), \( D(T) = 3 \) for \( r = 3 \), and \( 3 \leq D(T) \leq 4 \) for \( r \geq 4 \), see Fig 2 (in this figure and in the following ones the vertices are drawn in different levels according to their distances to \( X \)).

![Figure 2: Tree contained in C.](image)

\[ V(T) = \{v_1, v_2, \ldots, v_{r+1}\} \] with \( v_1 = z \), \( v_2 = t \), then \( d(v_i, X) = \mu \) for \( i = 1, 2 \), and \( d(v_i, X) = \mu - 1 \) for \( i = 3, \ldots, r + 1 \). Since \( \mu = \max\{d(u, X) : u \in V(C)\} \), then \( S_i^- = \emptyset \) for \( i = 1, 2 \), and \( N_i^+ = \emptyset \) for \( i = 3, \ldots, r + 1 \). Moreover, \( S_i^- = \emptyset \) for \( i = 1, 2 \), and \( N_i^+ = \emptyset \) for \( i = 3, \ldots, r + 1 \), because by hypothesis \( G[\mu(X) \cap V(C)] \) does not contain any edge. We have \( |\mu_i-((S_i^-) \cap X)| \geq |S_i^-| \) for \( i = 1, 2 \); and \( |\mu_i-((S_i^-) \cap X)| \geq |S_i^-| \), \( |\mu_i-((S_i^-) \cap X)| \geq |S_i^-| \) and \( |\mu_i-((N_i^-) \cap X)| \geq |S_i^+| \) for \( i = 3, \ldots, r + 1 \). Furthermore, the sets \( \mu_i-((S_i^-) \cap X) \) (\( i = 1, 2 \), and \( \mu_i-((S_i^-) \cap X) \), \( \mu_i-((N_i^-) \cap X) \) and \( \mu_i-((N_i^-) \cap X) \) (\( i = 3, \ldots, r + 1 \) are mutually disjoint, because otherwise, a cycle of length at most \( 2(\mu - 1) + 2D(T) \leq g - 1 \) exists, an absurdity. Therefore,

\[
|X| \geq \sum_{i=1}^{2} |\mu_i-((S_i^-) \cap X)| + \sum_{i=3}^{r+1} \left( |\mu_i-((S_i^-) \cap X)| + |\mu_i-((N_i^-) \cap X)| \right)
\]

\[ \geq \sum_{i=1}^{r+1} |S_i^-| + \sum_{i=3}^{r+1} \left( |S_i^-| + |S_i^+| \right) = \sum_{i=1}^{r+1} d(v_i) - 2r \geq \xi_r(G), \]

a contradiction. Hence we continue the proof by supposing that \( d_C(x, y) \geq 3 \) for every two vertices \( x, y \) in \( \mu(X) \cap V(C) \). Three cases are considered next.

**Case 1.** Suppose that \( \mu \geq 3 \).

Let \( u \) be any vertex of \( \mu(X) \cap V(C) \) and take \( v \in N(u) \). Notice that \( G[N(u) \cup N(v)] \) is a subgraph of \( C \) because \( \mu \geq 3 \). Moreover, \( |N(u) \cup N(v)| \geq r + 1 \), since \( \delta \geq \lceil (r + 1)/2 \rceil \) and \( g \geq 4 \). As \( G[\mu(X) \cap V(C)] \) has no edges and \( d_C(x, y) \geq 3 \) for every \( x, y \in \mu(X) \cap V(C) \),
Thus, the set of vertices $V(T) = \{v_1, \ldots, v_{r+1}\}$ with $v_1 = u$, $v_2 = v$, is such that $d(v_1, X) = \mu$, $d(v_i, X) = \mu - 1$ for $i = 2, \ldots, s$, and $d(v_i, X) = \mu - 2$, for $i = s + 1, \ldots, r + 1$. Observe that for each $i = s + 1, \ldots, r + 1$, every neighbor of a vertex in $N_i^\mu$ must be at distance $\mu - 1$ from $X$, because there are no edges in $N_\mu(X) \cap V(C)$; moreover, as $\delta \geq 2$ and $g \geq 6$, we have that every vertex in $N_i^\mu$ has at least a neighbor not belonging to $S_i^\mu$, hence $|N(N_i^\mu) \cap S_i^\mu| \geq |N_i^\mu|$.

Now, any two sets of $N_{\mu-1}(S_i^-) \cap X$, $N_{\mu-2}(S_i^-) \cap X$, $N_{\mu-1}(S_i^\pm) \cap X$ and $N_{\mu-1}(N_i^-) \cap X$ (for $i = 2, \ldots, s$), $N_{\mu-2}(v_i) \cap X$, $N_{\mu-2}(S_i^-) \cap X$, $N_{\mu-1}(N_i^\mu) \cap X$, $N_{\mu-1}(N_i^-) \cap X$ and $N_{\mu-2}(N_i^-) \cap X$ (for $i = s + 1, \ldots, r + 1$) are disjoint, because otherwise a cycle of length at most $g - 1$ is found; for $r \geq 3$, a cycle of length at most $2\mu + 6 \leq 2(\lfloor (g - 5)/2 \rfloor - 1) + 6 \leq g - 1$; and for $r = 2$, as $S_2^+ = S_3^+ = 0$, a cycle of length at most $2\mu + 2 \leq 2(\lfloor (g - 1)/2 \rfloor - 1) + 2 \leq g - 1$. Likewise, $|N_{\mu-1}(S_i^-) \cap X| \geq |S_i^-|$, $|N_{\mu-2}(S_i^-) \cap X| \geq |S_i^-|$, $|N_{\mu-1}(S_i^-) \cap X| \geq |S_i^-|$ and $|N_{\mu-1}(N_i^-) \cap X| \geq |S_i^-|$ (for $i = 2, \ldots, s$), $|N_{\mu-2}(v_i) \cap X| \geq |S_i^-|$ and $|N_{\mu-2}(S_i^-) \cap X| \geq |S_i^-|$ (for $i = s + 1, \ldots, r + 1$).

Further, since $\delta \geq \lceil (r + 1)/2 \rceil \geq 2$ we get

$$|N_{\mu-1}(N_i^\mu) \cap S_i^\mu) \cap X| + |N_{\mu-1}(N_i^\pm) \cap X| + |N_{\mu-2}(N_i^-) \cap X| \geq |S_i^\mu| \text{ for } i = s + 1, \ldots, r + 1.$$

Hence,

$$|X| \geq |N_{\mu-1}(S_i^-) \cap X|$$

$$+ \sum_{i=2}^{s} |N_{\mu-2}(S_i^-) \cap X| + |N_{\mu-1}(S_i^\pm) \cap X| + |N_{\mu-1}(N_i^-) \cap X|$$

$$+ \sum_{i=s+1}^{r+1} |N_{\mu-2}(v_i) \cap X| + |N_{\mu-2}(S_i^-) \cap X|$$

$$+ \sum_{i=s+1}^{r+1} \left( |N_{\mu-1}(N_i^\mu) \cap S_i^\mu) \cap X| + |N_{\mu-1}(N_i^\pm) \cap X| + |N_{\mu-2}(N_i^-) \cap X| \right)$$

$$\geq |S_i^-| + \sum_{i=2}^{s} (|S_i^-| + |S_i^\pm| + |S_i^+|) + \sum_{i=s+1}^{r+1} (|S_i^-| + |S_i^\pm| + |S_i^+|)$$

$$= \sum_{i=1}^{r+1} d(v_i) - 2r \geq \xi_r(G),$$

against our assumptions.

**Case 2.** Suppose that $\mu = 2$ and no two vertices $x, y \in N_\mu(X) \cap V(C)$ such that $3 \leq d_C(x, y) \leq \lceil (r - 1)/2 \rceil + 1$ exist.
Let us take some vertex \( z \in N_\mu(X) \cap V(C) \). Since \( T_z = G[\{z\} \cup N(z)] \) is a star contained in \( C \), there exists in \( C \) a tree \( T' \) on \( r + 1 \) vertices with diameter at most \( 2 + \lfloor (r - 1)/2 \rfloor \) that contains \( T_z \), as can be seen in Fig 4 (a). The set of vertices \( V(T') = \{v_1, v_2, \ldots, v_{r+1}\} \) with \( v_1 = z \) is such that \( d(v_i, X) = 1 \), for \( i = 2, \ldots, r + 1 \). It is clear that \( S_i^+ = \emptyset \), for \( i = 1, \ldots, r + 1 \) and \( S_i^- = \emptyset \). Further, the sets \( N(S_1^+) \cap X \), and \( N(v_i) \cap X \) and \( N(S_i^-) \cap X \), for \( i = 2, \ldots, r + 1 \) are mutually disjoint. Hence,

\[
|X| \geq |N(S_1^+) \cap X| + \sum_{i=2}^{r+1} (|N(v_i) \cap X| + |N(S_i^-) \cap X|)
\]

\[
\geq |S_1^-| + \sum_{i=2}^{r+1} (|S_i^-| + |S_i^-|) = \sum_{i=1}^{r+1} d(v_i) - 2r \geq \xi_r(G),
\]

which is impossible.

**Case 3.** Suppose that \( \mu = 2 \) and \( 3 \leq d_C(x, y) \leq \lfloor (r - 1)/2 \rfloor + 1 \) for some two vertices \( x, y \in N_\mu(X) \cap V(C) \).

Let us choose a pair of vertices \( z, t \in N_\mu(X) \cap V(C) \) in such a way that \( d_C(z, t) \leq d_C(x, y) \) for every \( x, y \in N_\mu(X) \cap V(C) \). Let us consider in \( C \) a tree \( T' \) on \( r + 1 \) vertices with diameter at most \( 2 + \lfloor (r - 1)/2 \rfloor \), such that \( T' \) contains \( \{z, t\} \). If \( d_C(z, t) \leq \lfloor (r - 1)/2 \rfloor \), then we take \( T' \) as formed from the star \( T_z \), the shortest \((z, t)\)-path, and the star \( T_1 \) (see Fig 4 (b)); if \( d_C(z, t) = \lfloor (r - 1)/2 \rfloor + 1 \), then we consider tree \( T' \) as formed from the star \( T_z \) and the shortest \((z, t)\)-path (see Fig 4 (c)). The set of vertices \( V(T') = \{v_1, v_2, \ldots, v_{r+1}\} \) is then such that \( v_1 = z \), \( v_2 = t \) and \( d(v_j, X) = 1 \), for \( i = 3, \ldots, r + 1 \). Obviously, \( S_i^+ = \emptyset \) for \( i = 1, \ldots, r + 1 \) (because of the way \( z, t \) have been chosen), and \( S_i^- = \emptyset \) for \( i = 1, 2 \). Moreover, any two sets of \( N(S_i^+) \cap X \) (for \( i = 1, 2 \)) and \( N(v_i) \cap X \), \( N(S_i^+) \cap X \) (for \( i = 3, \ldots, r + 1 \)) are disjoint. Therefore,

\[
|X| \geq \sum_{i=1}^{2} |N(S_i^+) \cap X| + \sum_{i=3}^{r+1} (|N(v_i) \cap X| + |N(S_i^-) \cap X|)
\]

\[
\geq \sum_{i=1}^{r+1} (|S_i^-| + |S_i^-|) = \sum_{i=1}^{r+1} d(v_i) - 2r \geq \xi_r(G),
\]

an absurdity.

Having arrived at a contradiction in any case, we can conclude that \( \mu \geq \lfloor (g - 5)/2 \rfloor \) if \( r \geq 3 \) or \( \mu \geq \lfloor (g - 1)/2 \rfloor \) if \( r = 2 \).

**Lemma 2.4** Let \( r \geq 3 \) be an integer and let \( G \) be a \( \kappa_r \)-connected graph with odd girth \( g \geq r + 5 \) and minimum degree \( \delta \geq \lfloor (r + 1)/2 \rfloor \). Let \( X \) be a minimum \( P_r \)-cutset of \( G \), and assume that
there exists a component $C$ of $G - X$ such that $\max\{d(u, X) : u \in V(C)\} = (g - \delta)/2$. If $G$ is non $\kappa_r$-optimal, then there exists $u \in N_{(g-\gamma)/2}(X) \cap V(C)$ such that $|N_{(g-\gamma)/2}(u) \cap X| = 1$.

**Proof:** From item (i) of Lemma 2.2, we have $(g - \delta)/2 = \mu \geq 2$, hence $g \geq 11$. We reason by contradiction assuming that $|N_{(g-\gamma)/2}(X) \cap V(C)| \geq 2$ for any $w \in N_{(g-\gamma)/2}(X) \cap V(C)$. By applying Lemmas 2.2 (ii) and 2.3, we can take an edge $uv$ in $G[N_{(g-\gamma)/2}(X) \cap V(C)]$. Taking into account that $g \geq 11$, the subgraph $G'[\{u, v\} \cup N(u) \cup N(v)]$ is a tree contained in $C$. Moreover, as $\delta \geq [(r + 1)/2]$, we can choose in $G'[\{u, v\} \cup N(u) \cup N(v)]$ a tree $T$ of order $r + 1$, whose set of vertices $V(T) = \{v_1, v_2, \ldots, v_{r+1}\}$ is such that $v_1 = u$, $v_2 = v$, $v_3 \in (N(u) \cup N(v)) \cap N_{(g-\gamma)/2}(X) \cap V(C)$ for $i = 3, \ldots, s$, $v_i \in N(u) \setminus N_{(g-\gamma)/2}(X) \cap V(C)$ for $i = s + 1, \ldots, t$, and $v_i \in N(v) \setminus N_{(g-\gamma)/2}(X) \cap V(C)$ for $i = t + 1, \ldots, r + 1$. Without loss of generality, we may suppose that $\sum_{i=s+1}^{r+1} |N_i^-| \geq \sum_{i=t+1}^{r+1} |N_i^-|$. Then any two sets of $N_{(g-\gamma)/2}(S_i^-) \cap X$ and $N_{(g-\gamma)/2}(S_i^-) \cap X$ for $i = 1, \ldots, s$, $N_{(g-\gamma)/2}(S_i^-) \cap X$, $N_{(g-\gamma)/2}(N_i^-) \cap X$ and $N_{(g-\gamma)/2}(N_i^-) \cap X$ for $i = s + 1, \ldots, t$, and $N_{(g-\gamma)/2}(N_i^-) \cap X$ and $N_{(g-\gamma)/2}(N_i^-) \cap X$ for $i = t + 1, \ldots, r + 1$, are pairwise disjoint, because otherwise a cycle of length at most $2(g - \delta)/2 + 6 = g - 1$ would be found. Moreover, taking into account that $\delta \geq 2$, the assumption $|N_{(g-\gamma)/2}(w) \cap X| \geq 2$ for any $w \in N_{(g-\gamma)/2}(X) \cap V(C)$, and the lower bound for the girth of $G$, it follows that $|N_{(g-\gamma)/2}(S_i^-) \cap X| \geq |S_i^-|$ and $|N_{(g-\gamma)/2}(N_i^-) \cap X| \geq 2|S_i^-|$ for $i = 1, \ldots, s$, $|N_{(g-\gamma)/2}(v_i) \cap X| \geq |S_i^-|$, $|N_{(g-\gamma)/2}(S_i^-) \cap X| \geq |S_i^-|$, $|N_{(g-\gamma)/2}(N_i^-) \cap X| \geq 2|N_i^-|$, $|N_{(g-\gamma)/2}(N_i^-) \cap X| \geq |N_i^-|$ for $i = s + 1, \ldots, t$, and $|N_{(g-\gamma)/2}(v_i) \cap X| \geq |S_i^-|$, $|N_{(g-\gamma)/2}(S_i^-) \cap X| \geq |S_i^-|$ and $|N_{(g-\gamma)/2}(N_i^-) \cap X| \geq |N_i^-|$ for $i = t + 1, \ldots, r + 1$. Hence,

\[
|X| \geq \sum_{i=1}^{s} (|N_{(g-\gamma)/2}(S_i^-) \cap X| + |N_{(g-\gamma)/2}(S_i^+) \cap X|) + \sum_{i=s+1}^{r+1} |N_{(g-\gamma)/2}(v_i) \cap X| + \sum_{i=t+1}^{r+1} (|N_{(g-\gamma)/2}(v_i) \cap X| + |N_{(g-\gamma)/2}(S_i^-) \cap X| + |N_{(g-\gamma)/2}(N_i^-) \cap X|)
\]

\[
\geq \sum_{i=1}^{s} (|S_i^-| + 2|S_i^-|) + \sum_{i=s+1}^{r+1} (|S_i^-| + |S_i^-| + 2|N_i^-| + |N_i^-|)
\]

\[
+ \sum_{i=t+1}^{r+1} (|S_i^-| + |S_i^-| + |N_i^-|)
\]

\[
\geq \sum_{i=1}^{s} (|S_i^-| + |S_i^-|) + \sum_{i=s+1}^{r+1} (|S_i^-| + |S_i^-| + |N_i^-| + |N_i^-|)
\]

\[
\geq \sum_{i=1}^{r+1} (|S_i^-| + |S_i^-|) + \sum_{i=s+1}^{r+1} (|S_i^-| + |S_i^-| + |S_i^+|) = \sum_{i=1}^{r+1} d(v_i) - 2r \geq \xi_r(G),
\]

which is impossible.

Thus, there exists some $u \in N_{(g-\gamma)/2}(X) \cap V(C)$ such that $|N_{(g-\gamma)/2}(u) \cap X| = 1$.  

Lemma 2.5 Let $G$ be a $\kappa_2$-connected graph with even girth $g \geq 8$ and minimum degree $\delta \geq 3$. Let $X$ be a minimum $P_2$-cutset of $G$, and assume that there exists a connected component $C$ of $G - X$ such that $\max\{d(u, X) : u \in V(C)\} = (g - 4)/2$. If $G$ is non $\kappa_2$-optimal, then there exists $u \in N_{(g-4)/2}(X) \cap V(C)$ such that $|N_{(g-4)/2}(u) \cap X| = 1$.

Proof: First, let us see that $G[N_{(g-4)/2}(X) \cap V(C)]$ contains a path of length two. Following Lemmas 2.2 (ii) and 2.3, we know that $G[N_{(g-4)/2}(X) \cap V(C)]$ contains edges. Suppose that $G[N_{(g-4)/2}(X) \cap V(C)]$ does not contain any path of length two. Let $v_1v_2$ be an isolated edge in $G[N_{(g-4)/2}(X) \cap V(C)]$ and consider the path $v_1v_2v_3$ in $C$, where $v_3 \in N(v_2) - v_1$. Clearly, $d(v_3, X) = (g - 6)/2$. In this case, the sets $N_{(g-6)/2}(S_i^-) \cap X$ for $i = 1, 2, N_{(g-6)/2}(v_3) \cap X$, $N_{(g-6)/2}(S_3^-) \cap X$, and $N_{(g-6)/2}(N_3^-) \cap X$ are mutually disjoint, since there are no cycles of length at most $2(g - 4)/2 + 3 = g - 1$. Further, again $|N_{(g-6)/2}(S_i^-) \cap X| \geq |S_i^-|$ for $i = 1, 2, 3$, $|N_{(g-6)/2}(v_3) \cap X| \geq |S_3^-|$ and $|N_{(g-6)/2}(N_3^-) \cap X| \geq |S_3^-|$; moreover, we have $|N_{(g-6)/2}(N_3^-) \cap X| \geq |N_3^-| \geq |S_3^-|$, the second inequality following from $\delta \geq 3$ and the fact that every vertex in $S_3^+$ can have at most one neighbor belonging to $N_3^+$ (since we are assuming that no paths of length two exist in $G[N_{(g-4)/2}(X) \cap V(C)]$). Hence, as $S_1^+ = S_1^- = S_2^+ = S_2^- = \emptyset$, we deduce

$$|X| \geq \sum_{i=1}^2 |N_{(g-6)/2}(S_i^-) \cap X| + |N_{(g-6)/2}(v_3) \cap X|$$

$$+ |N_{(g-6)/2}(S_3^+) \cap X| + |N_{(g-6)/2}(S_3^-) \cap X|$$

$$\geq |S_1^-| + |S_2^-| + |S_3^-| + |S_3^+| = d(v_1) + d(v_2) + d(v_3) - 4$$

$$\geq \xi_2(G),$$

an absurdity. Hence, $G[N_{(g-4)/2}(X) \cap V(C)]$ contains some path $v_1v_2v_3$. Without loss of generality, we may suppose that $|S_i^-| \geq |S_3^-|$. Then the sets $N_{(g-4)/2}(v_i) \cap X$ for $i = 1, 2, 3$, and $N_{(g-4)/2}(S_i^-) \cap X$ for $i = 1, 2$ are mutually disjoint, because cycles of length at most $2(g - 4)/2 + 3 = g - 1$ are forbidden. Likewise, we have $|N_{(g-4)/2}(v_i) \cap X| \geq |S_i^-|$ for $i = 1, 2, 3$, and also $|N_{(g-4)/2}(S_i^-) \cap X| \geq 2|S_i^-|$ for $i = 1, 2$ by hypothesis. Then

$$|X| \geq \sum_{i=1}^3 |N_{(g-4)/2}(v_i) \cap X| + \sum_{i=1}^2 |N_{(g-4)/2}(S_i^-) \cap X|$$

$$\geq \sum_{i=1}^3 |S_i^-| + \sum_{i=1}^2 2|S_i^-|$$

$$\geq \sum_{i=1}^3 (|S_i^-| + |S_i^-|)$$

$$= \sum_{i=1}^3 d(v_i) - 4$$

$$\geq \xi_2(G),$$

contradicting that $G$ is non $\kappa_2$-optimal. Hence, there is a vertex $u \in N_{(g-4)/2}(X) \cap V(C)$ such that $|N_{(g-4)/2}(u) \cap X| = 1$. ■

The core of the paper is given next in the following two theorems, which are a consequence of all above results.
Theorem 2.1 Let \( r \geq 2 \) be an integer. Let \( G \) be a \( \kappa_r \)-connected graph with girth \( g \geq r + 5 \), minimum degree \( \delta \geq \lceil (r + 1)/2 \rceil \) and diameter \( D \). Then \( G \) is \( \kappa_r \)-optimal if any of the following assertions holds:

(i) \( D \leq g - 7 \), for \( r \geq 3 \).

(ii) \( D \leq g - 4 \), for \( r = 2 \) and \( \delta \geq 3 \).

(iii) \( D \leq g - 4 \), for \( r = 2 \) and \( g \) odd.

(iv) \( D \leq g - 5 \), for \( r = 2 \) and \( g \) even.

Proof: (i) Suppose that \( G \) is non-\( \kappa_r \)-optimal and consider two connected components, \( C \) and \( C' \) of \( G - X \) where \( X \) is a minimum \( P_r \)-cutset. Then, by Lemmas 2.2 and 2.3 we have that there exists a vertex \( u \in V(C) \) such that \( d(u, X) \geq \lfloor (g - 6)/2 \rfloor \) and there exists a vertex \( u' \in V(C') \) such that \( d(X, u') \geq \lfloor (g - 6)/2 \rfloor \). Hence, \( g - 7 \geq D \geq d(u, X) + d(X, u') \geq 2 \lfloor (g - 6)/2 \rfloor \), which is only possible when \( g \) is odd and \( D = g - 7 \). In this case, we deduce that

\[
\max \{d(v, X) : v \in V(C)\} = \max \{d(v', X) : v' \in V(C')\} = (g - 7)/2.
\]

Now, by applying Lemma 2.4, we can choose \( u_0 \in N_{[g-7]/2}(X) \cap V(C) \) and \( u'_0 \in N_{[g-7]/2}(X) \cap V(C') \) in such a way that \( N_{[g-7]/2}(u_0) \cap X = \{x_0\} \) and \( N_{[g-7]/2}(u'_0) \cap X = \{y_0\} \). This fact allows us to deduce that \( x_0 = y_0 \). Notice that \( |N(u_0) \cap N_{[g-7]/2}(X)| = d(u_0) - 1 \), otherwise we would have \( |N_{[g-7]/2}(u_0) \cap X| \geq 2 \) which is impossible. As \( \delta \geq 2 \) we can take any \( v \in N(u_0) \cap N_{[g-7]/2}(X) \cap V(C) \). Clearly, \( d(v, u'_0) = g - 7 \), \( d(v, X) = (g - 7)/2 \) and \( d(u'_0, X - x_0) \geq (g - 5)/2 \), hence \( d(v, x_0) = (g - 7)/2 \). But then, the shortest \( (u_0, x_0) \)-path of length \( (g - 7)/2 \), the shortest \( (v, x_0) \)-path of length \( (g - 7)/2 \) and the edge \( u_0v \) form a cycle of length at most \( g - 6 \), a contradiction. Therefore, item (i) holds. Items (ii), (iii) and (iv) are proved similarly, by applying Lemmas 2.2 and 2.3, plus Lemma 2.5 for (ii).

Observe that the equalities \( \kappa_2(G) = \xi_2(G) \) (for above item (ii)) and \( \kappa_r(G) = \xi_r(G) \) (for item (i), except if \( \delta = 2 \)) follow when the corresponding constraint on the diameter is satisfied, because of Lemma 2.1. Next, we provide other sufficient conditions for \( \kappa_r \)-optimality in \( \kappa_r \)-connected graphs. These conditions are formulated in terms of the periphery of the graph, \( \text{Per}(G) \), which is the subgraph of \( G \) induced by its peripheral vertices, that is, by the vertices having eccentricity equal to the diameter.

Theorem 2.2 Let \( r \geq 2 \) be an integer. Let \( G \) be a \( \kappa_r \)-connected graph with minimum degree \( \delta \geq \max\{3, \lceil (r + 1)/2 \rceil \} \) and girth \( g \geq r + 5 \). Then \( G \) is \( \kappa_r \)-optimal if \( \text{Per}(G) \) does not contain any edge and one of the following assertions holds:

(i) \( D = g - 6 \), for odd girth and \( r \geq 3 \).

(ii) \( D = g - 3 \), for even girth and \( r = 2 \).

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Proof: (i) Suppose that $G$ is non-$\kappa_r$-optimal and consider a minimum $P_r$-cutset $X$. Taking into account that the diameter of $G$ is equal to $g - 6$ and Lemmas 2.2 and 2.3, it follows for every component $C$ of $G - X$ that
\[
(g - 7)/2 \leq \mu(C) = \max\{d(u, X) : u \in V(C)\} \leq (g - 5)/2.
\]
First, assume that $\mu(C) = (g - 7)/2$ and $\mu(C') = (g - 5)/2$ for certain two components $C, C'$ of $G - X$. In this case, $C$ must contain some edge $uv$ such that $d(u, X) = d(v, X) = (g - 7)/2$ because of Lemmas 2.2 and 2.3. Let $u' \in V(C')$ such that $d(u', X) = (g - 5)/2$. We easily get that $d(u, u') = d(v, u') = g - 6$, that is, $uv$ is an edge contained in $Per(G)$ which contradicts the hypothesis. Second, assume that $\mu(C) = \mu(C') = (g - 7)/2$. Then, by Lemmas 2.2 and 2.3, both $G[N_{(g-7)/2}(X) \cap V(C)]$ and $G[N_{(g-7)/2}(X) \cap V(C')]$ contain some edge. By Lemma 2.4 there exists a vertex $u_0 \in V(C)$ such that $N_{(g-7)/2}(u_0) \cap X = \{x_0\}$ and there exists a vertex $u'_0 \in V(C')$ such that $N_{(g-7)/2}(u'_0) \cap X = \{x'_0\}$. As $|N(u_0) \cap N_{(g-7)/2}(X) \cap V(C)| = d(u_0) - 1 \geq 2$, we can take two vertices $v, w \in N(u_0) \cap N_{(g-7)/2}(X) \cap V(C)$. Obviously, $d(v, x'_0) \geq (g - 5)/2$ or $d(w, x'_0) \geq (g - 5)/2$, because otherwise the shortest $(v, x'_0)$-path of length $(g - 7)/2$, the shortest $(w, x'_0)$-path of length $(g - 7)/2$ and the path $wuw$ form a cycle of length at most $g - 5$, and this is impossible. So suppose, for instance, that $d(v, x'_0) \geq (g - 5)/2$. But this means that $g - 6 = D \geq d(v, u'_0) \geq \min\{d(v, x'_0) + d(x'_0, u'_0), d(v, X - x'_0) + d(X - x'_0, u'_0)\} \geq g - 6$, hence $d(v, u'_0) = g - 6 = D$ and $\{v, u'_0\} \subseteq V(Per(G))$. Analogously, there exist two vertices $v', w' \in N(u'_0) \cap N_{(g-7)/2}(X) \cap V(C')$. Reasoning as for $v, w$, we conclude that $\{v', u_0\} \subseteq V(Per(G))$. Hence, $\{u_0, v, u'_0, v'\} \subseteq V(Per(G))$ and therefore, $Per(G)$ contains some edge, against our assumptions. Item (ii) is shown in a similar way.  

3 Conclusions

For $r \geq 2$, Theorems 1.1 (items (i), (ii)), 2.1 and 2.2 assure suitable lower bounds for $\kappa_r(G)$ when the diameter $D(G)$ is appropriately upper bounded. Clearly, any of these results can be considered an improvement of some other when the constraint on the diameter is less restrictive or the lower bound for $\kappa_r(G)$ is larger, for similar values of the integer $r$ and of the minimum degree $\delta(G)$. In this regard, Theorem 2.2 can improve Theorem 2.1 for $\delta(G) \geq 3$ depending on the parity of the girth, provided that the periphery of the graph does not contain any edge. As far as the comparison of Theorem 2.1 with respect to Theorem 1.1 is concerned, one must first notice that $\kappa_r(G) \geq \xi_r(G)$ improves $\kappa_r(G) \geq (r + 1)\delta(G) - 2r$ as $\xi_r(G)$ can be quite larger than $(r + 1)\delta(G) - 2r$, especially for graphs $G$ with a high degree of non-regularity. Moreover, it is not difficult to see that Theorem 2.1 improves Theorem 1.1 (i) when $r \geq 5$ and $\delta(G) \geq \max\{3, \lceil(r + 1)/2\rceil\}$, since the constraint on the diameter for the former is less restrictive than that for the latter. Finally, the same kind of improvement is clear for Theorem 2.1 with respect to point (ii) of Theorem 1.1 when dealing with graphs with $\delta(G) \geq \max\{3, \lceil(r + 1)/2\rceil\}$, a little more restrictive constraint than $\delta(G) \geq \max\{3, \lceil(r - 1)/2\rceil\}$. 

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Acknowledgment

We would like to thank the referees for his/her helpful suggestions and comments.

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