On convexity in graphs

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Abstract

This is a kind of survey and glossary by Ignacio M Pelayo on convexity in graphs, including definitions, results, remarks...

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1 Abstract convexity

- Convexity space: A convexity space is an ordered pair \((V, C)\), where \(V\) is a non-empty set and \(C\) is a collection of \(V\)-subsets, to be regarded as convex sets, such that

(C1) \(\emptyset, V \in C\).
(C2) Arbitrary intersections of convex sets are convex.
(C3) Every nested union of convex sets is convex.

[Remark 1]: According to Gerard Sierksma (see [61]), in 1951 Levi [54] introduced the concept of convexity space as a pair \((X, C)\) with \(X\) a set and \(C\) a collection of subsets of \(X\) closed under intersections.

[Remark 2]: According to M. Faber and E. Jamison (see [34]), an alignment on a finite set \(X\) is a family \(\mathcal{L}\) of subsets of \(X\) (to be considered convex sets) satisfying axioms (C1) and (C2). The pair \((X, \mathcal{L})\) is called an aligned space.

- Halfspace: A halfspace is a convex set, of which the complement is convex as well.

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• **Convex hull**: Given a convexity space \((V,\mathcal{C})\) and a set \(S \subset V\), the **smallest** convex set \([S]_\mathcal{C} \in \mathcal{C}\) containing \(S\) is called the convex hull of \(S\).

**Remark**: Convexity spaces are finitary: A fundamental property of the convex hull in a convexity space is to be finitary: If \(\mathcal{C}\) is a collection of \(V\)-subsets satisfying axioms (C1) and (C2), then Axiom (C3) (nested union property) is equivalent to the following property (see [26]):

\((C'3)\) If \(x \in [A]_\mathcal{C}\), then \(x \in [F]_\mathcal{C}\) for some finite \(A\)-subset \(F\).

• **Redundant set**: Given a convexity space \((V,\mathcal{C})\) a set \(A \subset V\) is called redundant (see [26]) when it is not empty and has the property:

\([A]_\mathcal{C} = \bigcup_{a \in A} [A - a]_\mathcal{C}\)

• **Carathéodory number**: The Carathéodory number \(c\) of a convexity space \((V,\mathcal{C})\) is the **smallest** integer (if it exists) such that for any subset \(S \subset V\) and any point \(p \in [S]_\mathcal{C}\), there is a subset \(F\) of \(S\) with \(|F| \leq c\) and \(p \in [F]_\mathcal{C}\). That is:

\[c = \min\{m : S \subset V, p \in [S]_\mathcal{C} \Rightarrow \exists F \subset S \text{ s.t. } |F| \leq m, p \in [F]_\mathcal{C}\}\]

In other words (see [59]), \(c\) is the minimum integer \(k\) such that for all \(X \subseteq V\),

\([X]_\mathcal{C} = \bigcup\{[S]_\mathcal{C} : S \subseteq X, |S| \leq c\}\).

**Remark 1**: Carathéodory’s Fundamental Theorem: Each point in the convex hull of a set \(S\) in \(R^n\) is in the convex combination of \(n + 1\) or fewer points of \(S\). In other words, the Carathéodory number of \(R^n\) (with the euclidean convexity) is \(c = n + 1\) (since this number is sharp).

**Remark 2**: Irredundant sets and the Carathéodory number: In any convexity space, the Carathéodory number equals the maximum cardinality of an irredundant set.

• **Helly number**: The Helly number \(h\) of a convexity space \((V,\mathcal{C})\) is the **smallest** integer (if it exists) such that every family of convex sets has a nonempty intersection whenever each subfamily of size \(h\) does. In other words, \(h\) is the **smallest** integer such that every family of convex sets with an empty intersection contains a subfamily of at most \(h\) members with an empty intersection. That is:

\[h = \min\{m : \{S_i\}_{i \in I} \in \mathcal{P}(\mathcal{C}), \bigcap_{i \in I} S_i = \emptyset \Rightarrow \exists \{S_{ij}\}_{j=1}^m \text{ s.t. } \bigcap_{j=1}^m S_{ij} = \emptyset\}\]

Or, as was proved by G. Sierksma in [59]:

\[h = \min\{m : A \subset V, |A| \geq m + 1 \Rightarrow \bigcap_{a \in A} [A - a]_\mathcal{C} \neq \emptyset\}\]

Observe that in this definition the expression \(|A| \geq m + 1\) can be replaced by \(|A| = m + 1\).
Remark: Helly’s Theorem: If \( F \) is a family of more than \( n \) bounded closed convex sets in \( \mathbb{R}^n \), and if every \( n+1 \) members of \( F \) have at least one point in common, then all the members of \( F \) have at least one point in common. In other words, the Helly number of \( \mathbb{R}^n \) (with the euclidean convexity) is \( h = n + 1 \) (since this number is sharp).

- **Radon number**: The Radon number \( r \) of a convexity space \( (V, C) \) is the smallest integer (if it exists) such that every subset \( S \subset V \) with \( |S| \geq r \) admits a Radon partition, i.e., a (disjoint) partition \( S = S_1 \cup S_2 \) with \( [S_1]_C \cap [S_2]_C \neq \emptyset \). That is:

\[
  r = \min \{ m : A \subset V, |A| \geq m \Rightarrow \exists \{A_1, A_2\} \text{ s.t. } A_1 \cup A_2 = A, A_1 \cap A_2 = \emptyset, [A_1]_C \cap [A_2]_C \neq \emptyset \}.
\]

It was proved in [54] that \( r \geq h + 1 \) (see also [50]).

Remark: Radon’s Theorem: Any set \( V \) of (at least) \( n+2 \) points in \( \mathbb{R}^n \) can always be partitioned in two subsets \( V_1 \) and \( V_2 \) such that the convex hulls of \( V_1 \) and \( V_2 \) intersect. In other words, the Radon number of \( \mathbb{R}^n \) (with the euclidean convexity) is \( r = n + 2 \) (since this number is sharp).

- **Graph convexity space**: A graph convexity space is an ordered pair \( (G, C) \), formed by a connected graph \( G \), with vertex set \( V \), and a convexity \( C \) on \( V \) such that \( (V, C) \) is a convexity space satisfying the additional axiom:

\[(C4)\] Every member of \( C \) induces a connected subgraph of \( G \).

- **Separation axioms**: We shall consider the following separation axioms for convexity spaces (see [64], pg 53):

\( S_0 \): For every two distinct points there exists a convex which contains exactly one of them.

\( S_1 \): Every one-point subset is convex.

\( S_2 \): Any two distinct points are separated by halfspaces.

\( S_3 \): If \( S \) is a convex set and \( x \in S \), then there is a halfspace \( H \) with \( S \subset H \) and \( x \notin H \) ([64]).

\( S_3 \): Every convex set is an intersection of halfspaces ([4]). (?)

\( S_4 \): Any two disjoint convex sets are separated by halfspaces.

- **Convex independency**: A subset \( A \subset V \) of a convexity space \( (V, C) \) is called convexly independent if \( a \notin [A - a]_C \) for every \( a \in A \). Observe that every convex dependent set is redundant, since for every \( a \in A \), \( a \in [A - a]_C \iff [A]_C = [A - a]_C \). Notice also that every subset of a convexly independent set is convexly independent.

- **Free set**: A subset \( A \subset V \) of a convexity space \( (V, C) \) is called free if it is both convex and convexly independent. Observe that a set \( A \) is free iff all its subsets are convex.
**Remark**: Free sets and the Helly number: In any convexity space, the Helly number is always at least as large as the cardinality of its largest free set. Moreover, if the convexity space satisfies the antixchange property (i.e. it is a convex geometry), then these two numbers are equal (see [46]).

- **Helly independency**: A subset $A \subset V$ of a convexity space $(V, C)$ is called Helly independent if:
  \[
  \bigcap_{a \in A} [A - a]_C = \emptyset
  \]

  Notice that every Helly independent set is a convex independent set, since for every $b \in A$, $b \in [A - b]_C \iff b \in \bigcap_{a \in A} [A - a]_C$.

**Remark**: Helly dependent sets and the Helly number: In any convexity space, the Helly number is the smallest integer $h$ such that every $(h + 1)$-element set is Helly dependent.

- **Interval convexities**: Let $V$ be a set and $I : V \times V \rightarrow 2^V$ be a mapping such that $x, y \in I(x, y)$ for every $x, y \in V$. The $I$-closed subsets of $V$ are subsets $C \subseteq V$ such that $I(C \times C) = C$. The set $\mathcal{C}_I$ of $I$-closed subsets satisfies axioms (C1), (C2) and (C3) of convexity spaces. The function $I$ is called an interval-function of the convexity space $(V, \mathcal{C}_I)$. Convexity spaces admitting an interval-function are named interval-convexity spaces (see [26]).

**Remark 1**: Redundant sets and the Carathéodory number: In an interval-convexity space, the Carathéodory number is the smallest integer $c$ such that every $(c + 1)$-element set is redundant. As a consequence, in any interval-convexity space, the cardinalities of irredundant sets form an interval $[0, c]$ of $\mathbb{N}$.

**Remark 2**: Counterexample: The previous property fails for general convexity spaces. For instance the convexity of a set $V$ constituted by all subsets of cardinality $\leq n$ has Charathéodory number $n + 1$ but every $k$-element set is redundant for $2 \leq k \leq n$.

**Remark 3**: Helly independent sets and the Helly number: In an interval-convexity space, the Helly number equals the maximum cardinality of an independent set (see [7], Sec. 5), (check whether the interval condition is needed ).
2 Path convexities in graphs

• **Distance:** The distance $d(u, v)$ between two vertices $u$ and $v$ in a connected graph $G$ is the length of a shortest $u - v$ path in $G$.

• **Geodesic:** A $u - v$ path of length $d(u, v)$ is called a $u - v$ geodesic.

• **Monophonic path:** A $u - v$ path is called monophonic (or induced path, or chordless path, or minimal path, or $m$-path) if it contains no chords (i.e., edges joining two non-consecutive vertices of the path). In other words, a path $P$ is called monophonic iff $\langle V(P) \rangle_G = P$.

• **Triangle path:** Chords of a path giving rise to triangles are called short chords of the path. A path which allows only short chords is called a triangle path or simply a $t$-path.

• **Path (graph) convexities:** The most natural convexities in a graph are path convexities (a type of interval convexity) defined by a system $\mathcal{P}$ of paths in $G$ that contain all geodesics. The canonical choices for $\mathcal{P}$ are provided by selecting all paths, triangle paths, minimal paths and geodesics. For example, the minimal path convexity (also called monophonic convexity, see [34]) is defined as follows: a subset of vertices $S$ is $m$-convex if it contains for any $x, y \in S$ all minimal paths between $x$ and $y$ (see [4, 26]). The triangle convexity (that is, the $t$-convex sets) is similarly defined (see [14]). The intrinsic or standard convexity is the geodesic convexity (see [9]).

• **Metric convexities:** K. Menger [Ergebnisse Math. Kolloq. Wien 1 (1931), 20–27; Jbuch 57, 756] introduced the so-called metric convexity in a metric space $(X, d)$. Namely, he called a set $A \subseteq X$ convex if $[x, y] := \{ z \in X: d(x, z) + d(z, y) = d(x, y) \} \subseteq A$ holds for all $x, y \in A$ (see [63]). Certainly, the geodesic convexity in graphs is an example of metric convexity.

• **$g_k$-convexity:** Let $k$ be an integer. A subset of vertices $S$ is $g_k$-convex if it contains for any $x, y \in S$ such that $d(x, y) \leq k$ all geodesics between $x$ and $y$ (see [35]).

**Remark:** Notice that:

geodesic $\Rightarrow$ m-path $\Rightarrow$ t-path

And hence:

$t$-convex $\Rightarrow$ m-convex $\Rightarrow$ g-convex $\Rightarrow$ $g_k$-convex

In consequence:
\[ [A]_{gk} \subset [A]_g \subset [A]_m \subset [A]_t \]

Where, for instance, \([A]_m\) denotes the m-convex hull of \(A\).

## 3 Geodesic Convexity

### 3.1 Geodesics and Intervals

- **Closed interval**: For vertices \(u\) and \(v\) in a graph \(G\), the closed interval \(I[u, v]\) consists of \(u\) and \(v\) together with all vertices lying in a \(u - v\) geodesic. That is:

\[
I[u, v] = \{V(\rho) : \rho \text{ is a } u - v \text{ geodesic}\} = \{x \in V(G) : d(u, v) = d(u, x) + d(x, v)\}
\]

- **Geodetic closure**: For \(S \subset V(G)\), \(I[S]\) is the union of all closed intervals \(I[u, v]\) with \(u, v \in S\), and it is called the (geodetic) closure of \(S\). That is:

\[
I[S] = \bigcup_{u,v \in S} I[u, v]
\]

- **Geodetic iteration number**: The process of taking closures starting from a vertex set \(S\) can be repeated to obtain a sequence \(S^1, S^2, \ldots\) of geodetic closures, where

\[
S^0 = S, \quad S^1 = I[S], \quad S^2 = I[I[S]] = I^2[S], \ldots, \quad S^k = I[S^{k-1}] = I^k[S], \ldots
\]

Since \(V(G)\) is finite, the process must terminate with some smallest \(r\) for which \(S^r = S^{r+1}\). So, the value of \(r\) is called the geodetic iteration number of \(S\), \(gin(S)\).

As expected, the geodetic iteration number \(gin(G)\) (see [57]) of a graph \(G\), is defined as the maximum value of \(gin(S)\) over all \(S \subset V(G)\).

### 3.2 Convexity Number

- **Convex set**: A set \(S\) is convex if \(I[S] = S\). It is also called geodesically convex and distance convex. For example, the empty set, all singletons, all edges and \(V\) are (geodesically) convex. All of them are usually called trivial convex sets.

- **Convex hull**: The convex hull \([S]\) (also called \(|S|\)-polytope when \(S\) is finite) is the smallest convex set containing \(S\). That is, \([S] = S^r\), where \(r = gin(S)\). Observe that:

\[
S \subseteq I[S] \subseteq I^2[S] \subseteq \ldots \subseteq I^r[S] = [S] \subseteq V
\]

Certainly, \(S\) is convex iff \(S = I[S] = [S]\).
• **Convexity number**: The convexity number \( \text{con}(G) \) is the maximum cardinality of a proper convex set of \( V(G) \).

• **\((k,n)\)-graph**: A nontrivial connected graph \( G \) of order \( n \) and convexity number \( \text{con}(G) = k \) is called a \((k,n)\)-graph (see [19]).

• **\(H\)-Convex graph**: For a nontrivial connected graph \( H \), a connected graph \( G \) is an \( H \)-convex graph if \( G \) contains a maximum convex set \( S \) whose induced subgraph is \( \langle S \rangle = H \) (see [19]). Observe that, if \( H \) is not \( G \), then \( G \) is a \((|H|,n)\)-graph.

### 3.3 Hull Number

• **Hull set**: A set \( S \) of vertices of \( G \) is a hull set if \( [S] = V(G) \).

• **Hull number**: A hull set of \( G \) of minimum cardinality is a minimum hull set and its cardinality is the hull number \( h_n(G) \) (see [30]).

• **Hull vertex**: A vertex \( v \) of \( G \) is a hull vertex if it belongs to every hull set in \( G \).

• **Hull graph**: A graph \( G \) with a unique minimum hull set is called a hull graph.

### 3.4 Geodetic Number

• **Geodetic set**: A set \( S \) of vertices of \( G \) for which \( I[S] = V(G) \) is called a geodetic set (or a geodetic cover) for \( G \). Since \( I[S] \subseteq [S] \), every geodetic set is a hull set.

• **Geodetic number**: The geodetic number \( g_n(G) \) is the minimum cardinality of a geodetic set ([41]). Certainly, \( g_n(G) \geq h_n(G) \).

• **Upper geodetic number**: The upper geodetic number \( g_n^+(G) \) is the maximum cardinality of a minimal geodetic set of \( G \). Hence, \( h_n(G) \leq g_n(G) \leq g_n^+(G) \).

• **Edge-Geodetic set**: A set \( S \) of vertices of \( G \) for which every edge of \( G \) lies on some shortest path joining two vertices of \( S \) is called a edge-geodetic set for \( G \). Certainly, every edge-geodetic set is (vertex) geodetic.
3.5 Forcing sets

- **Forcing set**: For a minimum geodetic set $S$ [resp. minimum hull set] of $G$, a subset $T$ of $S$ with the property that $S$ is the unique minimum geodetic set [resp. minimum hull set] containing $T$ is called a forcing subset of $S$.

- **Forcing geodetic number**: The forcing geodetic number $f(S, g)$ of a minimum geodetic set $S$ is the minimum cardinality of a forcing subset for $S$, while the forcing geodetic number $f(G, g)$ of $G$ is the smallest forcing number among all minimum geodetic sets of $G$.

- **Forcing hull number**: The forcing hull number $f(S, h)$ of a minimum hull set $S$ is the minimum cardinality of a forcing subset for $S$, while the forcing hull number $f(G, h)$ of $G$ is the smallest forcing number among all minimum hull sets of $G$.

3.6 Extreme vertices

- **Simplicial vertex**: A vertex $v$ of a graph $G$ is a simplicial vertex if the subgraph induced by its neighborhood is a clique. That is, if the subgraph induced by $N[v] = v \cup N(v)$ is a clique. That is, if it appears in exactly one clique of the graph $G$. A clique containing at least one such simplicial vertex of $G$ is called a simplex of $G$.

- **Extreme vertex**: A vertex $v$ of a graph $G$ is an extreme vertex if $V - v$ is convex (see [30]). Notice that, both for $g$-convexity and $m$-convexity, extreme and simplicial vertex definitions are equivalent. As expected, $Ext(G)$ denotes the set of all extreme vertices of $G$. Notice that every geodetic set $S$ must contain the set $Ext(G)$.

- **Extreme order**: The extreme order $ex(G)$ of a graph $G$ is the number of extreme vertices in $G$.

- **Simple vertex**: A vertex $v$ of a graph $G$ is simple if the set $\{N[u] : u \in N[v]\}$ is totally ordered by inclusion ([31]). Observe that every simple vertex is simplicial.

- **Minkowsky-Krein-Milman property**: A graph $G$ is said to satisfy the Minkowsky-Krein-Milman property (or alternatively, is called either an MKM-graph, or a convex geometry, or an antimatroid) if every convex set is the hull set of its extreme vertices. Certainly, every MKM-graph is a hull graph. It has been proved that the unique MKM-graphs are the Ptolemaic graphs (see [34]), that is to say, the geodesic convexity (also called alignment) in every ptolemaic graph is a convex geometry (also called antimatroid)(see [25]).

**Remark**: Antiexchange property: In any aligned space, the MKM property is equivalent to the so-called Antiexchange property.: For any convex set $S$ and two distinct vertices $x, y \not\in S$, $x \in [S \cup \{y\}] \Rightarrow y \not\in [S \cup \{x\}]$ (see [25]).
• **Extreme geodesic graph:** A graph $G$ is an extreme geodesic graph if $gn(G) = ex(G)$, that is, if every vertex lies on a $u-v$ geodesic for some pair $u, v$ of extreme vertices. Clearly, every extreme geodesic graph is an $MKM$-graph (is this claim true?).

• **Eccentricity of a vertex:** The eccentricity of a vertex $u \in V(G)$ is defined as $ecc(u) = \max\{d(u, v) : v \in V\}$.

• **Eccentric vertex:** Given two vertices $u, v \in V(G)$, the vertex $v$ is said to be an eccentric vertex for $v$ if $d(u, v) = ecc(u)$.

• **Contour vertex:** A vertex $v$ in $G$ is a contour vertex if $ecc(v) \geq ecc(u)$, for all $u \in N(v)$. It is easy to see that every extreme vertex is a contour vertex (see [11]).

• **Contour of a graph:** The contour of a graph $G$, denoted $Ct(G)$, is the set of all its contour vertices. Similarly is defined the contour $Ct(S)$ of a convex set $S$ of $G$. It is easy to see that $[Ct(S)] = S$ (see [11]).
4 Monophonic Convexity

- **Monophonic interval**: For vertices $u$ and $v$ in a graph $G$, the (closed) monophonic interval $J[u, v]$ consists of $u$ and $v$ together with all vertices lying in a $u - v$ chordless path. That is:

$$J[u, v] = \{V(\rho) : \rho \text{ is a } u - v \text{ chordless path}\}$$

- **Monophonic closure**: For $S \subseteq V(G)$, $J[S]$ is the union of all monophonic intervals $J[u, v]$ with $u, v \in S$, and it is called the monophonic closure of $S$. That is:

$$J[S] = \bigcup_{u,v \in S} J[u, v]$$

- **m-convex set**: A set $S$ is monophonically convex (see [34]), or simply m-convex, if $J[S] = S$. For example, the empty set, all singletons, all edges and $V$ are m-convex. Certainly, every m-convex set is g-convex.

- **m-convex hull**: The m-convex hull $[S]_m$ is the smallest m-convex set containing $S$. Certainly, $S$ is m-convex iff $S = J[S] = [S]_m$.

- **Monophonic set**: A set $S$ of vertices of $G$ for which $J[S] = V(G)$ is called a monophonic set for $G$. Observe that every geodetic set is monophonic.

- **Monophonic number**: The monophonic number $mn(G)$ is the minimum cardinality of a monophonic set. Certainly, $mn(G) \leq gn(G)$.

- **Edge-monophonic set**: A set $S$ of vertices of $G$ for which every edge of $G$ lies on some induced path joining two vertices of $S$ is called an edge-monophonic set for $G$. Certainly, every edge-monophonic set is (vertex) monophonic and every edge-geodetic set is edge-monophonic.
5 Steiner Problems

- **Steiner tree**: For a connected graph $G$ of order $n \geq 3$ and a set $W \subset V(G)$, a tree $T$ contained in $G$ is a Steiner tree with respect to $W$ (or simply an Steiner $W$-tree) if $T$ is a tree of minimum order with $W \subset V(T)$.

- **Steiner distance**: For a connected graph $G$ of order $n \geq 3$ and a set $W \subset V(G)$, the Steiner distance $d_S(W)$ of $W$ is the size of a Steiner $W$-tree (see [56]).

- **Steiner spanning set**: Given a connected graph $G$ of order $n \geq 3$, a set $W \subset V(G)$ is called a Steiner spanning set of $G$ if $d_S(W) = n - 1$ (see [56]). In this paper, it is shown that every connected graph has a unique Steiner spanning set.

- **Steiner interval**: The Steiner interval $S(W)$ of a vertex set $W$ consists of all vertices in $G$ that lie on some Steiner tree with respect to $W$ (see [52]). That is:

  \[ S(W) = \{ w \in V(G) : w \text{ lies on a Steiner } W - \text{tree} \} \]

  Observe that if $|W| = 2$, then: $S(W) = I[W]$.

- **$k$-intersection interval**: Let $W$ a vertex $l$-set and $k \leq l$. The $k$-intersection interval $S_k(W)$ of $W$ is the intersection of all Steiner intervals of $k$-subsets of $W$ (see [52]). That is:

  \[ S_k(W) = \bigcap_{U \subset W, |U| = k} S(U) \]

- **Steiner set**: A vertex set $W$ is a Steiner set for $G$ if $S(W) = V(G)$ (see [20]). Notice that every Steiner spanning set is a Steiner set.

- **Steiner convex hull**: Given a vertex set $W$, the set $[S(W)]$ is called the Steiner convex hull of $W$.

  - **(Chartrand) Steiner number**: The minimum cardinality among the Steiner sets of $G$ is the Steiner number $st(G)$ (see [20]).

  - **(Oellerman) Steiner number**: The minimum cardinality among the Steiner spanning sets of $G$ is the Steiner number $s(G)$ (see [56]). Certainly, $st(G) \leq s(G)$.

  - **(Oellerman) Steiner sequence**: If $G$ is a connected graph of order $n$ and $k$ is an integer with $0 \leq k \leq n - 1$, then the $k$th Steiner number $s_k(G)$ of $G$ is the smallest positive integer $l$ for which there exists a set $S$ of $l$ vertices of $G$ such that $d(S) = k$. The sequence $s_0(G), s_1(G), \ldots, s_{n-1}(G)$ is called the Steiner sequence of $G$. Steiner sequences for trees are characterized in [56].
- **Edge Steiner set**: The set $S_e(W)$ of a vertex set $W$ consists of all edges in $G$ that lie on some Steiner tree with respect to $W$. The set $W$ is a Steiner set for $G$ if $S_e(W) = E(G)$. Certainly, every edge Steiner set is a Steiner set.

- **Edge Steiner number**: The minimum cardinality among the edge Steiner sets of $G$ is the edge Steiner number $st_e(G)$. Certainly, $st(G) \leq st_e(G)$.

### 6 Other Definitions

- **Local convexities**: For any notion of convexity on the vertex set of a graph $G$, at least four degrees of local convexity may be distinguished:

  1. $N(v)$ is convex for every vertex $v$ of $G$.
  2. $N^j(v)$ is convex for every vertex $v$ of $G$ and every $j > 0$.
  3. $N(K)$ is convex for every convex set $K$ of $G$.
  4. $N^j(K)$ is convex for every convex set $K$ of $G$ and every $j > 0$.

  Observe that conditions 3. and 4. are equivalent. It is was proved in [34] that all of these conditions are equivalent for m-convexity and hold if and only if the graph is chordal. The g-convexity case has been approached in [35].

- **Well-bridged cycle**: A cycle $C$ of a graph $G$ is well-bridged if, for each vertex $p$ of $C$, either the two neighbors of $p$ on $C$ are adjacent, or there is a bridge from $p$ to another vertex of $C$ (see [35]). Note that every 3-cycle is trivially well-bridged. It was proved in [35] that in a bridged graph every cycle is well-bridged.

- **Minimum geodetic subgraph**: A graph $F$ is a minimum geodetic subgraph if there exists a graph $G$ containing $F$ as an induced subgraph such that $V(F)$ is a minimum geodetic set for $G$ (see [17]).

- **Uniform set**: A set $S$ of vertices in a graph $G$ is uniform if the distance between every two distinct vertices of $S$ is the same fixed number.

- **Essential set**: A geodetic set $S$ is essential if for every two distinct vertices $u, v \in S$, there exists a third vertex $w$ of $G$ that lies in some $u - v$ geodesic but in no $x - y$ geodesic for $x, y \in S$ and $\{x, y\} \neq \{u, v\}$.

- **Polyconvex set**: A connected graph $G$ is polyconvex if for every integer $i$ with $1 \leq i \leq con(G)$, there exists a convex set of cardinality $i$ in $G$.

- **Convex Ramsey number**: for positive integers $s, t$, the convex Ramsey number $cr(s, t)$ is defined as the smallest positive integer $n$ such that for every graph $G$ of order $n$, either $G$ contains a maximum convex set of cardinality $\geq s$ or $\overline{G}$ contains a maximum convex set of cardinality $\geq t$.

- **Diametral path**: A diametral path $\rho$ is a geodesic such that $|\rho| = D$. 


• **Median procedure:** For a given set of vertices $S = \{x_1, x_2, \ldots, x_k\}$, of a finite connected graph, a median of $S$ is a vertex $x$ for which $\sum_{i=1}^{k} d(x_i, x)$ is minimum. The median procedure $\text{Med}$ is the function whose domain is $\mathcal{P}(V)$, defined by:

$$\text{Med}(S) = \{x : x \text{ is a median of } S\}.$$ 

• **Convex product space:** Let $(X_1, \mathcal{C}_1), (X_2, \mathcal{C}_2)$ be convexity spaces. Their convex product space is the pair $(X_1 \times X_2, \mathcal{C}_1 \oplus \mathcal{C}_2)$ where $\mathcal{C}_1 \oplus \mathcal{C}_2 = \{A \times B \mid A \in \mathcal{C}_1, B \in \mathcal{C}_2\}$.

• **Exchange property:** A convexity space $(X, \mathcal{C})$ satisfies the so-called exchange property (see [54, 61]) iff for each finite set $A$ in $X$ and each $p$ in $[A]_\mathcal{C}$, the following holds:

$$[A]_\mathcal{C} = \bigcup_{a \in A} [A - a + p]_\mathcal{C}$$

• **Exchange function:**

• **Exchange set:** A subset $A$ of a convexity space $(X, \mathcal{C})$ is called an exchange set if, for every $p \in X$,

$$[A]_\mathcal{C} \subset \bigcup_{a \in A} [A - a + p]_\mathcal{C}$$

Certainly, every redundant set is an exchange set.

• **Exchange number:** The exchange number $e$ of a convexity space $(V, \mathcal{C})$ is the smallest integer $k$ such that every set of cardinality at least $k$ is an exchange set.

**Remark:** Redundant sets and exchange number: Certainly, $e \leq c + 1$, since every redundant set is an exchange set.

• **Join-hull commutativity condition:** A convexity space $(X, \mathcal{C})$ is said to satisfy the join-hull commutativity condition if, for every $S \subseteq \mathcal{C}$ and for every $a \in X$,

$$[S \cup a]_\mathcal{C} = \bigcup_{s \in S} [s, a]_\mathcal{C}.$$ 

The label $\text{JHC}$ stand as the abbreviation for *Join-hull Commutativity*. Notice that a convex structure is JHC iff for each non-empty finite set $F$ and for each $a \notin [F]_\mathcal{C}$,

$$[F \cup a]_\mathcal{C} \subseteq \bigcup_{x \in [F]_\mathcal{C}} [x, a]_\mathcal{C}.$$ 

• **Cone-union condition:** A convexity space $(X, \mathcal{C})$ is said to satisfy the cone-union condition if, for every $S, S_1, \ldots, S_m \subseteq \mathcal{C}$ such that $S \subseteq \bigcup_{i=1}^{m} S_i$, and for every $a \in X$,

$$[S \cup a]_\mathcal{C} \subseteq \bigcup_{i=1}^{m} [S_i \cup a]_\mathcal{C}.$$
The label CUP stand as the abbreviation for Cone-union Property. Observe that JHC implies CUP, and notice that both properties are equivalent for convex structures with finite polytopes (see [64]).

- **Continuity property:**

- **Basis of a set:** Let $A$ be a subset of a convexity space $(X,C)$. A basis of $A$ is a minimal set $B \subseteq A$ such that $[B]_C = [A]_C$. Observe that every minimal hull set is a basis of $X$.

**Remark:** Basis and the Antiexchange property: The antiexchange property is equivalent to the property that every set $A \subseteq X$ has a unique basis.

7 Some classes of graphs

- **HHD-free graphs:** An HHD-free graph is any graph which contain no house, hole, or domino as an induced subgraph (see [25]).

- **Bridge:** A bridge in a cycle $C$ of a graph $G$ is a shortest path in $G - C$ joining nonconsecutive vertices of $C$ which is shorter than both of the paths in $C$ joining those vertices (see [25]).

- **Bridged graphs:** A graph is bridged if it contains no isometric cycles of length greater than 3. In other words, if every cycle $C$ of length at least 4 has a bridge (see [32, 33]). Notice that every wheel $W_{1,p}$ with $p \geq 6$ is bridged.

- **1-chord and 2-chord triangles:** An induced triangle $\Delta$ of a cycle is called 1-chord (resp. 2-chord) triangle if exactly one edge (resp. two edges) of the triangle $\Delta$ is a chord (resp. are chords).

- **Chordal graphs:** Are those without induced cycles of length greater than 3. In other words, if every cycle of length at least 4 has a chord, i.e., an edge joining nonconsecutive vertices in the cycle. Certainly, every chordal graph is a bridged graph. It is also clear that a graph is chordal if it is both bridged and HHD-free.

**Remark 1:** Simplicial vertices in chordal graphs: Every chordal graph has a simplicial vertex and, if it is not a clique, then it has two nonadjacent simplicial vertices.

**Remark 2:** Perfect elimination order characterization: A graph $G = (V,E)$ is chordal if and only if it has a perfect elimination order; that is, if there is a labelling of the vertex set of $G$, $V = \{v_1, v_2, \ldots, v_n\}$ such that, for every $i \in \{1, 2, \ldots, n\}$, $v_i$ is a simplicial vertex of the subgraph of $G$ induced by $\{v_i, v_{i+1}, \ldots, v_n\}$.

**Remark 3:** 1-chord triangle characterization: A graph is chordal iff every $k$-cycle with $k \geq 4$ has an induced 1-chord triangle ([24]).

**Remark 4:** Being chordal is a hereditary property: A graph is chordal iff all its induced subgraphs are chordal.

**Remark 5:** Intersection graph characterization: A graph is chordal iff it can be represented as the intersection graph of a family of subtrees of a tree ([37]).
• **Strongly chordal graphs**: A graph $G$ is said to be strongly chordal if it is chordal and every even cycle $C$ of length at least 6 in $G$ contains a strong chord, i.e., a chord joining two vertices whose distance on $C$ is odd ([31]). This chordal graph subclass is interesting since there are several combinatorial graph problems which are NP-hard in chordal graphs but polynomially solvable in strongly chordal graphs (see [24]).

**Remark 1**: A forbidden induced subgraphs characterization: A graph is strongly chordal iff it is chordal and $k$-sun free, for every $k \geq 3$ (see [31]).

**Remark 2**: 2-chord triangle characterization: A graph is strongly chordal iff it has no chordless 4-cycle and every $k$-cycle with $k \geq 5$ has an induced 2-chord triangle ([24]).

**Remark 3**: Simple elimination ordering characterization: A graph is strongly chordal iff it admits a simple elimination ordering, that is, if there is a labelling of the vertex set of $G$, $V = \{v_1, v_2, \ldots, v_n\}$ such that, for every $i \in \{1, 2, \ldots, n\}$, $v_i$ is a simple vertex of the subgraph of $G$ induced by $\{v_i, v_{i+1}, \ldots, v_n\}$.

• **Distance-hereditary graphs**: A distance-hereditary graph is a graph in which every induced path is a geodesic. These graphs have been introduced by Edward Howorka in [44]. In this paper, it was proved that a graph is distance-hereditary if and only if each cycle on at least five vertices has at least two crossing chords. As a consequence, the author easily derived that these graphs are perfect. Moreover, it was proved in [5] that these graphs are precisely the HHD-free graphs without 3-fans (i.e. $P_4 \times K_1$). Finally, D’Atri and Moscarini proved in [23] that distance-hereditary graphs correspond exactly to graphs in which all minimal connected subgraphs joining a given set of vertices have the same order (in other words, the distance-hereditary property can be extended to an arbitrary set of vertices).

• **Ptolemaic graphs**: A connected graph $G$ is Ptolemaic provided that for any four vertices $u_i$, $1 \leq i \leq 4$, of $G$, the six distances $d_{ij} = d_G(u_i, u_j)$, $i \neq j$, satisfy the inequality $d_{12}d_{34} \leq d_{13}d_{24} + d_{14}d_{23}$ (see [45]). In this paper, the author proved that the Ptolemaic graphs are the chordal distance-hereditary graphs. That is to say, a Ptolemaic graph is a chordal graph which contain no 3-fan as an induced subgraph. In other words, a ptolemaic graph is a chordal graph in which every 5-cycle has at least 3 chords. Notice that every Ptolemaic graph is strongly chordal. Observe also that not every Ptolemaic is an interval graph.

• **Weakly chordal graphs**: A graph $G$ is said to be weakly chordal if it has neither holes nor antiholes where a hole is an induced cycle on five or more vertices and an antihole is the complement of a hole.

• **Distance convex simple graphs**: A graph is called distance convex simple (d.c.s.), if it has no other (geodesically) convex sets than the trivial ones (see [43]). Notice that distance convex simple graphs are nothing but $(2, n)$-graphs.

• **Geodetic graphs**: A graph $G$ is geodetic if each pair of nodes in $G$ is joined by a unique shortest path.

• **Intersection graph**: Let $S$ be a set and $F = \{S_1, S_2, \ldots, S_p\}$ a nonempty family of distinct nonempty subsets of $S$ whose union is $\bigcup_{i=1}^p S_i = S$. The intersection graph of $F$ is denoted $\Omega(F)$ and defined by $V(\Omega(F)) = F$, with and $S_i$ and $S_j$ adjacent whenever $i \neq j$ and $S_i \cap S_j \neq \emptyset$. It is
well known that any finite graph can be realized as the intersection graphs of a family of convex bodies in the 3-dimensional Euclidean vector space $\mathbb{R}^3$, but not so in $\mathbb{R}^2$ or $\mathbb{R}$. (see [51]).

- **Asteroidal triple:** A 3-set of vertices of a connected graph $G$ is called an asteroidal triple (or simply an AT) if for any two of them, there is a path joining them that does not intersect the neighborhood of the third vertex (see [22]). For example, $\{a, b, c\}$ is an asteroidal triple of the split graph: $G = (V, E)$, $V = \{1, 2, 3, a, b, c\}$, $E = \{12, 23, 31, a1, b2, c3\}$. Note that this graph is Ptolemaic.

- **Interval graph:** A graph $P$ is an interval graph provided that one can assign to each $v \in V(P)$ an (open) interval $I_v \subset \mathbb{R}$ such that $I_u \cap I_v$ is nonempty precisely when $uv \in E(P)$. In other words, an interval graph is the intersection graph of a nonempty family of intervals (i.e. convex bodies) of the real line. Such graphs have been characterized in various ways, but the corresponding problem relative to $\mathbb{R}^2$ is still open (in 1971, but nowadays?) (see [51]).

  **Remark:** Interval graphs characterization: $G$ is an interval graph iff it is a chordal and AT-free ([53]).

- **Split graph:** A graph $G$ is called a split graph if both $G$ and its complementary $\bar{G}$ are chordal.

  **Remark:** Split graphs characterization: A connected graph $G$ is a split graph iff $V(G)$ can be partitioned into a disjoint union $X \cup Y$ where the induced subgraph $\langle X \rangle_G$ is complete and the induced subgraph $\langle Y \rangle_G$ has no edges (see [36]). For example, $k$-suns are split graphs.

  **Remark:** Split graphs as intersection graphs: A connected graph $G$ is a split graph iff it is the intersection graph of a collection of subtrees of a star.

- **Circular arc graph:** A circular arc graph is the intersection graph of a family of arcs of the circle. For example, cycles and interval graphs are two subclasses of circular arc graphs.

- **Directed path graph:** A directed path graph is the intersection graph of a collection of directed paths in a rooted directed tree.

- **Undirected path graph:** An undirected path graph is the intersection graph of a collection of paths of a tree. Notice that directed path graphs are clearly also undirected path.

- **Catval graph:** A catval graph is the intersection graph of a collection of subtrees of a caterpillar. Observe that every split graph is a catval graph.

  **Remark:** Catval graphs characterization: A graph $G = (V, E)$ is a catval graph if and only if $G' \setminus \{v \in V \mid v$ is simplicial$\}$ is an interval graph.

- **Isometric subgraph:** An isometric subgraph of a graph $G$ is an induced subgraph in which any two vertices have the same distance as in $G$. For example, every geodesic in a graph $G$ is an isometric subgraph of $G$. Observe also that a cycle has a bridge iff it is not isometric.

- **Median graphs:** A median graph $G = (V, E)$ is a graph satisfying the following property: $|I[u, v] \cap I[v, w] \cap I[w, u]| = 1$ for all distinct $u, v, w \in V$. In other words, graphs for which the 2-intersection intervals of every 3-set consist of a unique vertex (see [52]). As expected, the unique vertex comprising this intersection is called the median of the set $\{u, v, w\}$. 


• **Strongly geodetic graphs**: If each pair of nodes $u, v$ of a graph $G$ is joined by at most one path of length not more than the diameter $D(G)$, then it is called strongly geodetic. Clearly, every strongly geodetic graph is geodetic (but not conversely).

• **Perfect graph**: A graph $G$ is called perfect iff for every induced subgraph $H$ of $G$, the size of the largest clique $\omega(H)$ equals the chromatic number $\gamma(H)$ (i.e., the fewest number of colors necessary to color the vertices of $H$ so that adjacent vertices have different colors). For example, it is a known fact that every chordal graph is perfect.

• **Parity graph**: A parity graph is a graph in which all maximal independent sets have the same parity (even or odd). Parity graphs generalize well-covered graphs. Both bipartite and distance-hereditary graphs belong to this family.

• **Well-covered graph**: A well-covered graph is a graph in which every maximal independent set is a maximum independent set, that is, if all maximal independent sets have the same cardinality (?); Plummer introduced the concept in a 1970 paper.

• **Cograph**: A cograph is a $P_4$-free graph. All of them are permutation graphs, and thus both AT-free and perfect.

• **Simplicial graph**: A graph is called simplicial if every vertex of $G$ is a simplicial vertex or is adjacent to a simplicial vertex of $G$. It is known that the recognition problem of well-covered graphs (and of $Z_m$-well-covered graphs) in general is Co-NP-complete, but for some classes of graphs it is already polynomial (examples are given in [8]).
References


